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New tripartite entangled state generated by an asymmetric beam splitter and a parametric down-conversion amplifier

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Abstract

In three-mode Fock space we construct a new tripartite entangled state $|\alpha, \gamma\rangle_{\theta\lambda}$ which makes up a new quantum mechanical representation. The state $|\alpha, \gamma\rangle_{\theta\lambda}$ can be generated by using the set-up composed of an asymmetric beam splitter and a parametric down-conversion amplifier. We then show how to use $|\alpha, \gamma\rangle_{\theta\lambda}$ to find new squeezing operator and new squeezed state.

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1. Introduction

The concept of quantum entanglement is increasingly of interest in studies of quantum information and quantum communication. It was originated by Einstein, Podolsky and Rosen (EPR) in a paper arguing the incompleteness of quantum mechanics [1] and has played a key role in understanding some fundamental problems in quantum mechanics and quantum optics [2–7]. A beam splitter is perhaps the simplest tool to produce quantum entanglement. It is known that even one single-mode squeezed state incident on a beam splitter yields a bipartite entangled state [8]. In [9], Braunstein and Loock pointed out that continuous-variable quantum teleportation of arbitrary coherent states has been realized experimentally with bipartite entanglement built from two single-mode squeezed vacuum states combined at a beam splitter whose role is expressed by the operator B [6]. They also asserted that a sequence of beam splitter operations,

$$B_{N-1,N}(\pi/4)B_{N-2,N-1}(\cos^{-1}1/\sqrt{3})\times\cdots\times B_{1,2}(\cos^{-1}1/\sqrt{N}), \tag{1}$$

applied to one momentum squeezed vacuum mode 1 and $N-1$ position squeezed vacuum modes 2 through N, yields an N-mode state with N-party entanglement between all modes. They obtained the entangled N-mode state $\int \mathrm{d}x|x, x, \ldots, x\rangle$. This state is an eigenstate with total momentum zero and all relative positions $X_i - X_j = 0$ $(i, j = 1, 2, \ldots, N)$ [10].

All these references exhibit the role of the beam splitter in generating entanglement and entangled states. Meanwhile we know that a two-mode squeezed state (made of idler light and signal light) generated by parametric down-conversion process simultaneously is a two-mode entangled state in the frequency domain. It is natural to think if we combine the mechanism of both beam splitter (especially an *asymmetric beam splitter*) and parametric down conversion, then what kind of entangled states can be generated? We are motivated to construct an entangled state generated by an asymmetric beam splitter and a parametric down-conversion amplifier. On the other hand, we hope the newly explored entangled states could be qualified to make up a new quantum mechanical representation, so we need the explicit form of this kind of entangled states in Fock space. As Dirac pointed out in [11]: 'When one has a particular problem to work out in quantum mechanics, one can minimize the labor by using a representation in which the representatives of the more important abstract quantities occurring in that problem are as simple as possible', we believe that entangled state representations will be useful not only in treating many problems in quantum optics, but also can open up (explore) new research topics. In EPR's pioneer argument, the entanglement was due to the fact that two particles' relative positions $X_1 - X_2$ and their total momentum $P_1 + P_2$ can be simultaneously measured. Enlightened by EPR, in [12] the simultaneous eigenstate $|\eta\rangle$ of commutative operators $(X_1 - X_2, P_1 + P_2)$ in two-mode Fock space is found:

$$|\eta\rangle = \exp\left[-\tfrac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right]|00\rangle_{12}, \tag{2}$$

where $\eta = (\eta_1 + \mathrm{i}\eta_2)/\sqrt{2}$, $|00\rangle_{12}$ is the vacuum state, $\left(a_i, a_i^\dagger\right)$, $i = 1, 2$, are the Bose annihilation and creation operators, related to X_i and P_i by $X_i = \frac{1}{\sqrt{2}}\left(a_i + a_i^\dagger\right)$, $P_i = \frac{1}{\sqrt{2}\mathrm{i}}\left(a_i - a_i^\dagger\right)$. Experimentally, the $|\eta\rangle$ state can be generated as follows [13–16]: when a pair of incoming modes—one is the zero-momentum eigenstate $|p = 0\rangle_1 \sim \exp\left(\frac{1}{2}a_1^{\dagger 2}\right)|0\rangle_1$ (maximum squeezing in the p-direction) and the other is the zero-position eigenstate $|x = 0\rangle_2 \sim \exp\left(-\frac{1}{2}a_2^{\dagger 2}\right)|0\rangle_2$ (maximum squeezing in the x-direction)—impinge on a symmetric 50:50 beam splitter (without loss and phase shift), the outgoing state is a bipartite entangled state, i.e.,

$$\exp\left[-\frac{\pi}{4}\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)\right]|p = 0\rangle_1 \otimes |x = 0\rangle_2 = \exp\left[a_1^\dagger a_2^\dagger\right]|00\rangle. \tag{3}$$

Then making a local oscillator displacement $D(\eta) = \exp\left[\eta a_1^\dagger - \eta^* a_1\right]$ for $\exp\left[a_1^\dagger a_2^\dagger\right]|00\rangle$, the state $|\eta\rangle$ is obtained. However, when the beam splitter is not a 50:50 one, but an asymmetric one, then what is the output state when two light fields maximally squeezed in X_i and P_i, respectively, entering its two input ports and get superimposed? According to [17], the output state emerging from asymmetric beam splitter is defined by

$$\begin{aligned}|\eta\rangle_\theta = \exp\Big[&-\tfrac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^*\left(a_2^\dagger \sin 2\theta + a_1^\dagger \cos 2\theta\right) + \tfrac{1}{2}\eta^{*2}\cos 2\theta \\ &+ \tfrac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2}\right)\cos 2\theta + a_2^\dagger a_1^\dagger \sin 2\theta\Big]|00\rangle,\end{aligned} \tag{4}$$

where θ is related to the amplitude reflectivity and transmissivity of the asymmetric beam splitter, and a local oscillator displacement is also made. Clearly, when $\theta = \pi/4$, $|\eta\rangle_\theta$ reduces to $|\eta\rangle$. It is remarkable that $|\eta\rangle_\theta$ makes up a new complete set and is of importance from the quantum mechanics representation theory.

A question thus naturally arises: can we extend $|\eta\rangle_\theta$ state to the tripartite case in a direct way so that a new kind of tripartite entangled states of continuum variables can be constructed? The answer is affirmative. Our work is arranged as follows. In section 2, we briefly review the main properties of $|\eta\rangle_\theta$. In section 3, we introduce the new tripartite entangled state $|\alpha, \gamma\rangle_{\theta\lambda}$ which is the common eigenvector of three commutable operators, with λ being a squeezing parameter. In section 4, we discuss how to generate $|\alpha, \gamma\rangle_{\theta\lambda}$ by an asymmetric beam splitter and parametric down-conversion amplifier. In section 5, we investigate its properties, especially its completeness, partly non-orthogonal property and its Schmidt decomposition. In section 6, we show how to apply $|\alpha,\gamma\rangle_{\theta\lambda}$ to deriving new squeezing operator and generalized squeezed state.

2. The bipartite entangled state $|\eta\rangle_\theta$

We begin with briefly reviewing the properties of two-mode entangled state $|\eta\rangle_\theta$. $|\eta\rangle_\theta$ is the common eigenvector of commutative operators: $(X_2 - X_1\tan\theta)$ and $(P_1 + P_2\tan\theta)$, i.e.,

$$(X_2 - X_1\tan\theta)|\eta\rangle_\theta = -\eta_1\tan\theta|\eta\rangle_\theta, \tag{5}$$

$$(P_1 + P_2\tan\theta)|\eta\rangle_\theta = \eta_2\tan\theta|\eta\rangle_\theta. \tag{6}$$

This simultaneous measurement of $(X_2 - X_1\tan\theta)$ and $(P_1 + P_2\tan\theta)$ with accuracy is allowed by quantum mechanics and can be visualized in a generalized eight-port interferometer measurement [18].

Using the normal ordering form of $|00\rangle\langle 00| =: \exp\left\{-a_1^\dagger a_1 - a_2^\dagger a_2\right\}:$ and the technique of integration within an ordered product (IWOP) of operators [19, 20] we can smoothly prove the completeness relation

$$\sin 2\theta \int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle_{\theta\theta}\langle\eta| = 1, \tag{7}$$

so $|\eta\rangle_\theta$ make up a complete set. The overlap of $|\eta\rangle_\theta$ is

$${}_\theta\langle\eta'|\eta\rangle_\theta = 2\pi\delta(\eta_1 - \eta_1')\delta(\eta_2 - \eta_2')/\sin 2\theta, \qquad \eta = (\eta_1 + \mathrm{i}\eta_2)/\sqrt{2}. \tag{8}$$

According to Dirac's representation theory in quantum mechanics, the set of $|\eta\rangle_\theta$ makes up a new orthogonal and complete representation in the two-mode Fock space. The state $|\eta\rangle_\theta$ can be generated by an asymmetric beam splitter: operating the asymmetric beam splitter operator

$$S_2 = \exp\left[-\theta\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)\right] \tag{9}$$

on a pair of incoming modes $|p=0\rangle_1 \otimes |x=0\rangle_2$, we have

$$\begin{aligned} S_2|p=0\rangle_1 \otimes |x=0\rangle_2 &\sim \exp\left[\tfrac{1}{2}\left(a_1^\dagger\cos\theta + a_2^\dagger\sin\theta\right)^2 - \tfrac{1}{2}\left(a_2^\dagger\cos\theta - a_1^\dagger\sin\theta\right)^2\right] \\ &\quad\times \exp\left[2\theta\left(a_2^\dagger a_1 - a_1^\dagger a_2\right)\right]|00\rangle \\ &= \exp\left[a_1^\dagger a_2^\dagger\sin 2\theta + \tfrac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2}\right)\cos 2\theta\right]|00\rangle = |\eta = 0\rangle_\theta. \end{aligned} \tag{10}$$

Then operating the displacement operator $D_1(\eta) \equiv \exp\left[\eta a_1^\dagger - \eta^* a_1\right]$ on equation (10) leads to equation (4), i.e.,

$$D_1(\eta)\exp\left[a_1^\dagger a_2^\dagger\sin 2\theta + \tfrac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2}\right)\cos 2\theta\right]|00\rangle = |\eta\rangle_\theta. \tag{11}$$

Experimentally, this displacement can be implemented by reflecting the light field of $|\eta = 0\rangle_\theta$ from a partially reflecting mirror (say 99% reflection and 1% transmission) and adding through the mirror a field that has been phase and amplitude modulated according to the value $\eta \equiv |\eta|\,\mathrm{e}^{\mathrm{i}\varphi}$.

3. The new tripartite entangled state

We now introduce a new kind of three-mode entangled state in Fock space

$$\begin{aligned}|\alpha,\gamma\rangle_{\theta,\lambda} = \operatorname{sec}h\lambda \exp\{&-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\gamma|^2 + \tfrac{1}{2}\alpha^{*2}\cos 2\theta + a_1^\dagger(\alpha - \alpha^*\cos 2\theta) \\ &- a_2^\dagger(\alpha^* \operatorname{sec}h\lambda \sin 2\theta + \gamma^* \tanh\lambda) + \gamma a_3^\dagger + a_1^\dagger a_2^\dagger \operatorname{sec}h\lambda \sin 2\theta \\ &+ a_2^\dagger a_3^\dagger \tanh\lambda + \tfrac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2}\operatorname{sec}h^2\lambda\right)\cos 2\theta\}|000\rangle,\end{aligned} \tag{12}$$

where $\alpha = \alpha_1 + \mathrm{i}\alpha_2$, $\gamma = \gamma_1 + \mathrm{i}\gamma_2$ are the two complex numbers, λ is a real number (later, one can see that it is a squeezing parameter characterizing a parametric down-conversion amplifier [21]). In particular, when $\lambda = 0$, $\operatorname{sec}h\lambda = 1$ and $\tanh\lambda = 0$, then equation (12) reduces to

$$\begin{aligned}|\alpha,\gamma\rangle_{\theta,0} &= \exp\{-\tfrac{1}{2}|\alpha|^2 + \alpha a_1^\dagger - \alpha^*\left(a_1^\dagger\cos 2\theta + a_2^\dagger \sin 2\theta\right) + \tfrac{1}{2}\alpha^{*2}\cos 2\theta \\ &\quad + a_1^\dagger a_2^\dagger \sin 2\theta + \tfrac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2}\right)\cos 2\theta\}\exp\left[-\tfrac{1}{2}|\gamma|^2 + \gamma a_3^\dagger\right]|000\rangle \\ &\equiv |\alpha\rangle_\theta \otimes |\gamma\rangle_3,\end{aligned} \tag{13}$$

where $|\alpha\rangle_\theta$ is the bipartite entangled state in a_1–a_2-mode (10) generated by an asymmetric beam splitter, while $|\gamma\rangle_3$ is the coherent state in a_3-mode. On the other hand, when $\theta = \pi/4$, equation (12) becomes

$$\begin{aligned}|\alpha,\gamma\rangle_{\frac{\pi}{4},\lambda} = \operatorname{sec}h\lambda \exp\{&-\tfrac{1}{2}(|\alpha|^2 + |\gamma|^2) + a_1^\dagger\alpha + \gamma a_3^\dagger \\ &+ a_2^\dagger\left(a_1^\dagger - \alpha^*\right)\operatorname{sec}h\lambda + a_2^\dagger\left(a_3^\dagger - \gamma^*\right)\tanh\lambda\}|000\rangle,\end{aligned} \tag{14}$$

this is a tripartite entangled state which can be generated by a symmetric beam splitter and parametric down-conversion amplifier. For a review of various applications of the entangled state representation of continuum variables we refer to [22]. Operating a_i on $|\alpha,\gamma\rangle_{\theta,\lambda}$ respectively yields

$$\left(a_3 - a_2^\dagger \tanh\lambda\right)|\alpha,\gamma\rangle_{\theta\lambda} = \gamma|\alpha,\gamma\rangle_{\theta\lambda}, \tag{15}$$

$$\left(a_1 - a_1^\dagger\cos 2\theta - a_2^\dagger \operatorname{sec}h\lambda\sin 2\theta\right)|\alpha,\gamma\rangle_{\theta\lambda} = (\alpha - \alpha^*\cos 2\theta)|\alpha,\gamma\rangle_{\theta\lambda}, \tag{16}$$

$$\begin{aligned}&\left(a_2 - a_1^\dagger \operatorname{sec}h\lambda\sin 2\theta + a_2^\dagger \operatorname{sec}h^2\lambda\cos 2\theta - a_3^\dagger\tanh\lambda\right)|\alpha,\gamma\rangle_{\theta\lambda} \\ &\quad = -(\alpha^*\operatorname{sec}h\lambda\sin 2\theta + \gamma^*\tanh\lambda)|\alpha,\gamma\rangle_{\theta\lambda}.\end{aligned} \tag{17}$$

By combining equations (16) and (17) we see

$$(X_1\tan\theta\operatorname{sec}h\lambda - X_2 + X_3\tanh\lambda)|\alpha,\gamma\rangle_{\theta\lambda} = \sqrt{2}\tanh\lambda(\alpha_1\tan\theta\operatorname{csc}h\lambda + \gamma_1)|\alpha,\gamma\rangle_{\theta\lambda}, \tag{18}$$

$$(P_1\cot\theta\operatorname{sec}h\lambda + P_2 + P_3\tanh\lambda)|\alpha,\gamma\rangle_{\theta\lambda} = \sqrt{2}\tanh\lambda(\alpha_2\cot\theta\operatorname{csc}h\lambda + \gamma_2)|\alpha,\gamma\rangle_{\theta\lambda}. \tag{19}$$

Note that the three operators $(X_1\tan\theta\operatorname{sec}h\lambda - X_2 + X_3\tanh\lambda)$, $(P_1\cot\theta\operatorname{sec}h\lambda + P_2 + P_3\tanh\lambda)$ and $a_3 - a_2^\dagger\tanh\lambda)$ make up a complete commutable operator set.

4. Implementation of $|\alpha,\gamma\rangle_{\theta\lambda}$

Supposing we have the two-mode asymmetric entangled state as in equation (10) and an extra vacuum state $|0\rangle_3$,

$$|\alpha = 0\rangle_\theta \equiv \exp\left[a_1^\dagger a_2^\dagger\sin 2\theta + \tfrac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2}\right)\cos 2\theta\right]|00\rangle_{12}\otimes|0\rangle_3. \tag{20}$$

On the basis of this state, we make the squeezing transformation [9]

$$S_{23}a_2^\dagger S_{23}^{-1} = a_2^\dagger\cosh\lambda - a_3\sinh\lambda, \tag{21}$$

by the two-mode squeezing operator $S_{23} = \exp\left[\lambda\left(a_2^\dagger a_3^\dagger - a_2 a_3\right)\right]$, which means we let $a_2^\dagger$-mode and $a_3^\dagger$-mode interact in a parametric down-conversion amplifier via a nonlinear process, then we obtain

$$\begin{aligned} &S_{23} \exp\left[a_1^\dagger a_2^\dagger \sin 2\theta + \tfrac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2}\right) \cos 2\theta\right]|000\rangle \\ &\quad = \sec h\lambda \exp\left[a_3^2 \mathfrak{A} + a_3 \mathfrak{B} + \mathfrak{C}\right] \exp\left[a_2^\dagger a_3^\dagger \tanh\lambda\right]|000\rangle, \end{aligned} \tag{22}$$

where

$$\mathfrak{A} \equiv -\tfrac{1}{2} \sinh^2\lambda \cos 2\theta, \tag{23}$$

$$\mathfrak{B} \equiv \left(a_2^\dagger \cosh\lambda \cos 2\theta - a_1^\dagger \sin 2\theta\right) \sinh\lambda, \tag{24}$$

$$\mathfrak{C} \equiv a_1^\dagger a_2^\dagger \cosh\lambda \sin 2\theta + \tfrac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2} \cosh^2\lambda\right) \cos 2\theta. \tag{25}$$

Using the completeness relation of coherent state $\int \frac{\mathrm{d}^2 z}{\pi} |z\rangle_{33}\langle z| = 1$ and the following formula

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp[\zeta|z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}] = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left[\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right], \tag{26}$$

whose convergent condition is either

$$\mathrm{Re}(\zeta + f + g) < 0, \qquad \mathrm{Re}\left(\frac{\zeta^2 - 4fg}{\zeta + f + g}\right) < 0, \tag{27}$$

or

$$\mathrm{Re}(\zeta - f - g) < 0, \qquad \mathrm{Re}\left(\frac{\zeta^2 - 4fg}{\zeta - f - g}\right) < 0, \tag{28}$$

we have

$$\begin{aligned} &\exp\left[a_3^2 \mathfrak{A} + a_3 \mathfrak{B}\right] \exp\left[a_3^\dagger a_2^\dagger \tanh\lambda\right] = \int \frac{\mathrm{d}^2 z}{\pi} \exp\left[a_3^2 \mathfrak{A} + a_3 \mathfrak{B}\right] |z\rangle_{33}\langle z| \exp\left[a_3^\dagger a_2^\dagger \tanh\lambda\right] \\ &\quad = \int \frac{\mathrm{d}^2 z}{\pi} \colon \exp\left[-|z|^2 + z\left(\mathfrak{B} + a_3^\dagger\right) + z^*\left(a_2^\dagger \tanh\lambda + a_3\right) + z^2 \mathfrak{A} - a_3^\dagger a_3\right] \colon \\ &\quad = \colon \exp\left[\left(\mathfrak{B} + a_3^\dagger\right)\left(a_2^\dagger \tanh\lambda + a_3\right) + \left(a_2^\dagger \tanh\lambda + a_3\right)^2 \mathfrak{A} - a_3^\dagger a_3\right] \colon, \end{aligned} \tag{29}$$

and then substituting equation (29) into equation (22) we can rewrite equation (22) as

$$\begin{aligned} (22) &\equiv \sec h\lambda \exp\left[\left(\mathfrak{B} + a_3^\dagger\right) a_2^\dagger \tanh\lambda + a_2^{\dagger 2} \mathfrak{A} \tanh^2\lambda + \mathfrak{C}\right]|000\rangle \\ &= \sec h\lambda \exp\left[a_1^\dagger a_2^\dagger \sec h\lambda \sin 2\theta + a_2^\dagger a_3^\dagger \tanh\lambda + \tfrac{1}{2}\left(a_1^{\dagger 2} - a_2^{\dagger 2} \sec h^2\lambda\right) \cos 2\theta\right]|000\rangle \\ &= |\alpha = 0, \gamma = 0\rangle_{\theta\lambda}. \end{aligned} \tag{30}$$

Then making a two-mode displacement $D_1(\alpha) D_3(\gamma)$ for $|\alpha = 0, \gamma = 0\rangle_{\theta\lambda}$ by two local oscillators, we make up the state $|\alpha, \gamma\rangle_{\theta\lambda}$,

$$D_1(\alpha) D_3(\gamma) |\alpha = 0,\ \gamma = 0\rangle_{\theta\lambda} = |\alpha, \gamma\rangle_{\theta\lambda}, \tag{31}$$

where $D_i(\alpha) = \exp\left[\alpha a_i^\dagger - \alpha^* a_i\right]$ and the relation $D_i(\alpha) a^\dagger D_i^{-1}(\alpha) = a_i^\dagger - \alpha^*$ is used. Thus, $|\alpha, \gamma\rangle_{\theta\lambda}$ can be generated by asymmetric beam splitter and parametric down-conversion amplifier.

5. Properties of $|\alpha, \gamma\rangle_{\theta\lambda}$

We now investigate the major properties of $|\alpha, \gamma\rangle_{\theta\lambda}$. We need to know what is the integration measure with which $|\alpha, \gamma\rangle_{\theta\lambda}$ can make up a complete set. Using the normally ordered form of the vacuum projection operator

$$|000\rangle\langle 000| =: \exp\left[-a_1^\dagger a_1 - a_2^\dagger a_2 - a_3^\dagger a_3\right]:, \tag{32}$$

where the symbol : : denotes normal ordering, we can employ the IWOP technique and equation (26) to prove the completeness relation of $|\alpha, \gamma\rangle_{\theta\lambda}$,

$$\begin{aligned}\frac{\sin 2\theta}{\sec h^2\lambda}\int\frac{d^2\alpha\, d^2\gamma}{\pi^2}|\alpha, \gamma\rangle_{\theta\lambda,\theta\lambda}\langle\alpha, \gamma| &= \sin 2\theta\int\frac{d^2\alpha\, d^2\gamma}{\pi^2}:\exp[-|\gamma|^2 + \gamma\zeta + \gamma^*\zeta^\dagger]\\ &\quad\times\exp\left[-|\alpha|^2 + \alpha\xi + \alpha^*\xi^\dagger + \frac{1}{2}(\alpha^{*2} + \alpha^2)\cos 2\theta + \kappa\right]:\\ &= :\exp\left[\zeta\zeta^\dagger + \frac{1}{\sin^2 2\theta}\left(\xi\xi^\dagger + \frac{\cos 2\theta}{2}(\xi^2 + \xi^{\dagger 2})\right) + \kappa\right]: =: \exp[0]: = 1\end{aligned} \tag{33}$$

where

$$\zeta \equiv a_3^\dagger - a_2\tanh\lambda, \tag{34}$$

$$\xi \equiv a_1^\dagger - a_1\cos 2\theta - a_2\sec h\lambda\sin 2\theta, \tag{35}$$

$$\begin{aligned}\kappa &\equiv \left(a_1^\dagger a_2^\dagger + a_1 a_2\right)\sec h\lambda\sin 2\theta + \left(a_2^\dagger a_3^\dagger + a_2 a_3\right)\tanh\lambda\\ &\quad+\frac{1}{2}\left[a_1^{\dagger 2} + a_1^2 - \left(a_2^{\dagger 2} + a_2^2\right)\sec h^2\lambda\right]\cos 2\theta - \sum_{i=1}^{3} a_i^\dagger a_i.\end{aligned} \tag{36}$$

Here, the factor $\sin 2\theta/\sec h^2\lambda$ is integration measures needed for the completeness relation. It must be pointed out that the integration $d^2\alpha\, d^2\gamma$ is two fold in complex variables, not three fold, this is because the state $|\alpha, \gamma\rangle_{\theta\lambda}$ is entangled among three modes, which reduces the folds of integration. Thus, we note that although $|\alpha, \gamma\rangle_{\theta\lambda}$ is defined in three-mode Fock space, because its three modes are mutually entangled, it spans a completeness relation with two-fold complex integration measure. Furthermore, in order to examine if $|\alpha, \gamma\rangle_{\theta\lambda}$ constitutes an orthogonal set or not, using equations (18) and (19) we evaluate the following matrix elements in the $|\alpha, \gamma\rangle_{\theta\lambda}$ state:

$$\begin{aligned}&{}_{\theta\lambda}\langle\alpha', \gamma'|(X_1\tan\theta\sec h\lambda - X_2 + X_3\tanh\lambda)|\alpha, \gamma\rangle_{\theta\lambda}\\ &\quad= \sqrt{2}\tanh\lambda(\alpha_1\tan\theta\csc h\lambda + \gamma_1)_{\theta\lambda}\langle\alpha', \gamma'.|\alpha, \gamma\rangle_{\theta\lambda}\\ &\quad= \sqrt{2}\tanh\lambda(\alpha_1'\tan\theta\csc h\lambda + \gamma_1')_{\theta\lambda}\langle\alpha', \gamma'.|\alpha, \gamma\rangle_{\theta\lambda},\end{aligned} \tag{37}$$

$$\begin{aligned}&{}_{\theta\lambda}\langle\alpha', \gamma'|(P_1\cot\theta\sec h\lambda + P_2 + P_3\tanh\lambda)|\alpha, \gamma\rangle_{\theta\lambda}\\ &\quad= \sqrt{2}\tanh\lambda(\alpha_2\cot\theta\csc h\lambda + \gamma_2)_{\theta\lambda}\langle\alpha', \gamma'.|\alpha, \gamma\rangle_{\theta\lambda}\\ &\quad= \sqrt{2}\tanh\lambda(\alpha_2'\cot\theta\csc h\lambda + \gamma_2')_{\theta\lambda}\langle\alpha', \gamma'.|\alpha, \gamma\rangle_{\theta\lambda},\end{aligned} \tag{38}$$

which lead to

$$\sqrt{2}\tanh\lambda[(\alpha_1 - \alpha_1')\tan\theta\csc h\lambda + (\gamma_1 - \gamma_1')]_{\theta\lambda}\langle\alpha', \gamma'.|\alpha, \gamma\rangle_{\theta\lambda} = 0, \tag{39}$$

$$\sqrt{2}\tanh\lambda[(\alpha_2 - \alpha_2')\cot\theta\csc h\lambda + (\gamma_2 - \gamma_2')]_{\theta\lambda}\langle\alpha', \gamma'.|\alpha, \gamma\rangle_{\theta\lambda} = 0, \tag{40}$$

thus we can see

$$ {}_{\theta\lambda}\langle\alpha',\gamma'|\alpha,\gamma\rangle_{\theta\lambda} \propto \frac{\coth^2\lambda}{2}\delta[(\alpha_1-\alpha_1')\tan\theta\,\mathrm{csc}\,h\lambda+\gamma_1-\gamma_1'] \times\delta[(\alpha_2-\alpha_2')\cot\theta\,\mathrm{csc}\,h\lambda+\gamma_2-\gamma_2']. \tag{41} $$

Equation (41) shows that the inner product of $|\alpha,\gamma\rangle_{\theta\lambda}$ involves two δ functions.

To explain in more detail why $|\alpha,\gamma\rangle_{\theta\lambda}$ is an entangled state, we make the following two-fold Fourier transformation:

$$ \int_{-\infty}^{\infty}\mathrm{d}\alpha_2\,\mathrm{d}\gamma_2|\alpha,\gamma\rangle_{\theta\lambda}\,\mathrm{e}^{\mathrm{i}(u\alpha_2+v\gamma_2)} = W(u,v,\alpha_1,\gamma_1)\left|\frac{1}{\sqrt{2}}(\alpha_1-u)\right\rangle_1\otimes\left|\frac{1}{\sqrt{2}}(-v+\gamma_1)\right\rangle_3 \otimes\left|-\frac{1}{\sqrt{2}}[(u+\alpha_1)\sec h\lambda\tan\theta+(v+\gamma_1)\tanh\lambda]\right\rangle_2, \tag{42} $$

where the three single-mode states all belong to the set of coordinate eigenvectors

$$ |q\rangle_i = \pi^{-1/4}\exp\left[-\tfrac{1}{2}q^2+\sqrt{2}qa_i^\dagger-\tfrac{1}{2}a_i^{\dagger 2}\right]|0\rangle_i, \tag{43} $$

and $W(u,v,\alpha_1,\gamma_1)$ is a normalization factor

$$ W = \frac{\sqrt{2}\sec h\lambda}{\pi^{-7/4}\cos\theta}\exp\left[-\tfrac{1}{4}[(v+\gamma_1)\sec h\lambda-(u+\alpha_1)\tan\theta\tanh\lambda]^2\right]. \tag{44} $$

Thus, the inverse transformation of equation (42) is

$$ |\alpha,\gamma\rangle_{\theta\lambda} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{d}v\,W\,\mathrm{e}^{-\mathrm{i}(u\alpha_2+v\gamma_2)}\left|\frac{1}{\sqrt{2}}(\alpha_1-u)\right\rangle_1\otimes\left|\frac{1}{\sqrt{2}}(\gamma_1-v)\right\rangle_3 \otimes\left|-\frac{1}{\sqrt{2}}[(u+\alpha_1)\sec h\lambda\tan\theta+(v+\gamma_1)\tanh\lambda]\right\rangle_2. \tag{45} $$

This is just the Schmidt decomposition of $|\alpha,\gamma\rangle_{\theta\lambda}$, so $|\alpha,\gamma\rangle_{\theta\lambda}$ is an entangled state [23]. In momentum representation, the Schmidt decomposition of $|\alpha,\gamma\rangle_{\theta\lambda}$ can be expressed as

$$ |\alpha,\gamma\rangle_{\theta\lambda} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{d}v\,W'\,\mathrm{e}^{-\mathrm{i}(u\alpha_1+v\gamma_1)}\left|\frac{1}{\sqrt{2}}(u+\alpha_2)\right\rangle_1\otimes\left|\frac{1}{\sqrt{2}}(v+\gamma_2)\right\rangle_3 \otimes\left|-\frac{1}{\sqrt{2}}[(u-\alpha_2)\sec h\lambda\cot\theta+(v-\gamma_2)\tanh\lambda]\right\rangle_2, \tag{46} $$

where the three single-mode states all belong to the set of momentum eigenvectors:

$$ |p\rangle_i = \pi^{-1/4}\exp\left[-\tfrac{1}{2}p^2+\sqrt{2}\mathrm{i}pa_i^\dagger+\tfrac{1}{2}a_i^{\dagger 2}\right]|0\rangle_i, \tag{47} $$

and $W'(u,v,\alpha_2,\gamma_2)$ is a normalization factor:

$$ W' = \frac{\sqrt{2}\sec h\lambda}{\pi^{-7/4}\sin\theta}\exp\left[-\tfrac{1}{4}[(v-\gamma_2)\sec h\lambda-(u-\alpha_2)\cot\theta\tanh\lambda]^2\right]. \tag{48} $$

The quantification of multipartite entanglement, even for pure states, is still the subject of current research [10]. In general, multipartite inseparability criteria cannot be formulated in such a compact form as those for bipartite. In order to verify genuine N-party entanglement, one has to rule out any possible partially separable forms. In principle, this can be done by considering all possible bipartite splittings or groupings and, for instance, applying the negative partial transpose criterion [24]. In order to see the entanglement involved in $|\alpha,\gamma\rangle_{\theta\lambda}$

more clearly, by tracing out one party, the remaining two-party state is

$$\begin{aligned}
\mathrm{Tr}_1|\alpha,\gamma\rangle_{\theta\lambda\theta\lambda}\langle\alpha,\gamma| &= \int\frac{\mathrm{d}^2z}{\pi}\,{}_1\langle z|\alpha,\gamma\rangle_{\theta\lambda\theta\lambda}\langle\alpha,\gamma|z\rangle_1 \\
&= \frac{\sec h^2\lambda}{\sin 2\theta}\colon\exp\left[-|\gamma|^2+\left(a_2^\dagger a_3^\dagger+a_2a_3-\gamma^*a_2^\dagger-\gamma a_2\right)\tanh\lambda\right] \\
&\quad\times\exp\left[\gamma a_3^\dagger+\gamma^*a_3-a_2^\dagger a_2\tanh^2\lambda-a_3^\dagger a_3\right]\colon
\end{aligned} \tag{49}$$

where we have inserted the completeness of coherent state $\int\frac{\mathrm{d}^2z}{\pi}|z\rangle_{11}\langle z|$ and used equation (26). Observing equations (49) one can see that, after tracing out one party, $a_2^\dagger$-mode is coupled with $a_3^\dagger$-mode. In other words, the remaining two-mode fields are still in the bipartite entangled state. Tracing out another mode will yield a similar result.

6. Application of the $|\alpha,\gamma\rangle_{\theta\lambda}$ representation

As an application of the $|\alpha,\gamma\rangle_{\theta\lambda}$ representation, we build the following ket–bra operator in an integration form:

$$U_\theta(\sigma,\lambda)\equiv\frac{\sin 2\theta}{\sec h^2\lambda}\int\frac{\mathrm{d}^2\alpha\,\mathrm{d}^2\gamma}{(\pi\mu)^2}|\alpha/\mu,\gamma/\mu\rangle_{\theta\lambda,\theta\lambda}\langle\alpha,\gamma|, \tag{50}$$

where $(\alpha,\gamma)\to(\alpha/\mu,\gamma/\mu)$ is a c-number dilation transformation. The meaning of discussing (50) lies in generating new squeezed state by an asymmetric beam splitter and parametric down-conversion amplifier. It should be pointed out that $U_\theta(\sigma,\lambda)$ is a new three-mode squeezing operator (for a review of squeezed states we refer to [25]). Using equation (12) and the IWOP technique we can directly perform the integral in equation (50):

$$\begin{aligned}
U_\theta(\sigma,\lambda) &= \frac{2\mu}{\sqrt{K}}\sin 2\theta\sec h\sigma\exp\left[\left(\frac{1}{2}-\frac{2}{K}(\mu^2+1-\alpha^{*2}-\mu^2\alpha^2\cos^2 2\theta)\right)a_1^{\dagger 2}\cos 2\theta\right] \\
&\quad\times\exp\left[\left(1+\frac{2}{K}(2\mu^2\alpha^2\cos^2 2\theta-\mu^2-1)\right)a_1^\dagger a_2^\dagger\sec h\lambda\sin 2\theta+a_2^\dagger a_3^\dagger\tanh\sigma\tanh\lambda\right] \\
&\quad\times\exp\left[a_2^{\dagger 2}\left(\frac{2\mu^2\alpha^2}{K}\sin^2 2\theta-\frac{1}{2}\right)\sec h^2\lambda\cos 2\theta\right]\colon\exp\left[-\left(a_1^\dagger a_1+a_2^\dagger a_2+a_3^\dagger a_3\right)\right] \\
&\quad\times\exp\left[\frac{2\mu}{K}\left[(1+\mu^2)(1+\cos^2 2\theta)-2\cos^2 2\theta\left(\alpha^{*2}+\mu^2\alpha^2\right)\right]a_1^\dagger a_1\right] \\
&\quad\times\exp\left[\frac{\mu}{K}\left[(1+\mu^2-2\alpha^{*2})a_1^\dagger a_2+(1+\mu^2-2\mu^2\alpha^2)a_2^\dagger a_1\right]\sec h\lambda\sin 4\theta\right] \\
&\quad\times\exp\left[\frac{2\mu(1+\mu^2)}{K}a_2^\dagger a_2\sec h^2\lambda\sin^2 2\theta+\left(a_3^\dagger a_3+a_2^\dagger a_2\tanh^2\lambda\right)\sec h\sigma\right]\colon \\
&\quad\times\exp\left[\left(\frac{1}{2}-\frac{2\mu^2}{K}(1+\mu^2-\mu^2\alpha^2-\alpha^{*2}\cos^2 2\theta)\right)a_1^2\cos 2\theta\right] \\
&\quad\times\exp\left[\left(\frac{2\mu^2\alpha^{*2}}{K}\sin^2 2\theta-\frac{1}{2}\right)a_2^2\sec h^2\lambda\cos 2\theta-a_2a_3\tanh\sigma\tanh\lambda\right] \\
&\quad\times\exp\left[a_1a_2\left(1-\frac{2\mu^2}{K}(1+\mu^2-2\alpha^{*2}\cos^2 2\theta)\right)\sec h\lambda\sin 2\theta\right],
\end{aligned} \tag{51}$$

where we have set $\mu = e^{\sigma}$ and $K = 1 + \mu^4 - 2\mu^2 \cos 4\theta$. Especially, when $\theta = \pi/4$, equation (51) reduces to

$$U_{\theta=\frac{\pi}{4}} = \sec h^2\sigma \exp\left[\left(a_1^\dagger a_2^\dagger \sec h\lambda + a_2^\dagger a_3^\dagger \tanh\lambda\right)\tanh\sigma\right]\exp\left[\left(a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3\right)\ln \sec h\sigma\right] \times \exp[-(a_1 a_2 \sec h\lambda + a_2 a_3 \tanh\lambda)\tanh\sigma], \quad (52)$$

which is just a new three-mode squeezing operator. Using equation (52), we see

$$\begin{aligned} U_{\theta=\frac{\pi}{4}} a_1 U^{-1}_{\theta=\frac{\pi}{4}} &= a_1 \cosh\sigma - a_2^\dagger \sec h\lambda \sinh\sigma, \\ U_{\theta=\frac{\pi}{4}} a_2 U^{-1}_{\theta=\frac{\pi}{4}} &= a_2 \cosh\sigma - \left(a_1^\dagger \sec h\lambda + a_3^\dagger \tanh\lambda\right)\sinh\sigma, \\ U_{\theta=\frac{\pi}{4}} a_3 U^{-1}_{\theta=\frac{\pi}{4}} &= a_3 \cosh\sigma - a_2^\dagger \tanh\lambda \sinh\sigma. \end{aligned} \quad (53)$$

By introducing the two quadratures

$$X = \frac{1}{\sqrt{2}}(X_1 \sec h\lambda - X_2 + X_3 \tanh\lambda), \quad (54)$$

$$P = \frac{1}{\sqrt{2}}(P_1 \sec h\lambda - P_2 + P_3 \tanh\lambda), \qquad [X, P] = \mathrm{i}, \quad (55)$$

we derive

$$U_{\theta=\frac{\pi}{4}} X U^{-1}_{\theta=\frac{\pi}{4}} = e^{\sigma} X, \qquad U_{\theta=\frac{\pi}{4}} P U^{-1}_{\theta=\frac{\pi}{4}} = e^{-\sigma} P, \quad (56)$$

which shows the standard squeezing for two mutually conjugate operators in an opposite way. Operating $U_{\theta=\frac{\pi}{4}}$ on the three-mode vacuum state $|000\rangle$, we have

$$U_{\theta=\frac{\pi}{4}}|000\rangle = \sec h^2\sigma \exp\left[\left(a_1^\dagger a_2^\dagger \sec h\lambda + a_2^\dagger a_3^\dagger \tanh\lambda\right)\tanh\sigma\right]|000\rangle, \quad (57)$$

which is a new three-mode squeezed vacuum state. The quantum fluctuation of the operator quadratures in the state $U_{\theta=\frac{\pi}{4}}|000\rangle$ is

$$\langle(\Delta X)^2\rangle = \tfrac{1}{2} e^{-2\sigma}, \qquad \langle(\Delta P)^2\rangle = \tfrac{1}{2} e^{2\sigma} \quad (58)$$

thus the minimum uncertainty relation still remains

$$\Delta X \Delta P \equiv \sqrt{\langle(\Delta X)^2\rangle\langle(\Delta P)^2\rangle} = \tfrac{1}{2}. \quad (59)$$

In summary, we have introduced a new tripartite entangled state $|\alpha, \gamma\rangle_{\theta\lambda}$ in three-mode Fock space. Such states are potentially useful, because they not only make up a complete representation [11], but also can be generated by an asymmetric beam splitter and parametric down-conversion amplifier. Using $|\alpha, \gamma\rangle_{\theta\lambda}$ we have derived new squeezing operator and squeezed state. Thus, we see again the intrinsic relation between entanglement and squeezed state.

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